J. Fluid Mech. (1971), *vol.* 48, *part* 2, *pp.* 209–228
Printed in Great Britain

On laminar boundary-layer blow-off. Part 2

By D. R. KASSOY
Mechanical Engineering Department, University of Colorado

(Received 25 June 1970 and in revised form 23 February 1971)

Several examples of incipient blow-off phenomena described by the compressible similar laminar boundary-layer equations are considered. An asymptotic technique based on the limit of small wall shear, and the use of a novel form of Prandtl's transposition theorem, leads to a complete analytical description of the blow-off behaviour. Of particular interest are the results for overall boundary-layer thickness, which imply that, for a given large Reynolds number, classical theory fails for a sufficiently small wall shear. A derivation of a new distinguished limit of the Navier–Stokes equations, the use of which will lead to uniformly valid solutions to blow-off type problems for $Re \to \infty$, is included. A solution for uniform flow past a flat plate with classical similarity type injection, based on the new limit, is presented. It is shown that interaction of the injectant layers and the external flow results in a favourable pressure gradient, which precludes the classical blow-off catastrophy.

1. Introduction

An interesting aspect of the study of flows with mass addition involves the phenomenon of blow-off. Here, under conditions which depend upon the specific configuration of the system being studied, it is possible to develop a region of vanishingly small wall shear, and ultimately observe a form of boundary-layer separation. In general, the occurrence of this condition is determined by the overall interaction of the mass flow distribution of injected fluid with the prescribed external flow (and hence the pressure gradients involved). The added fluid, assumed to be injected normal to the surface, can be turned toward the direction of the external flow by favourable pressure gradients and/or viscous forces. Should the totality of these effects become small near the surface, so that injected fluid streamlines are turned only slightly, the wall shear is small, and blow-off may be observed.

The prototype blow-off analysis, performed first by Schlichting & Bussmann (1943), but more definitively by Emmons & Leigh (1954), describes uniform flow past a flat plate with an injection distribution proportional to $x^{-\frac{1}{2}}$ in terms of classical boundary-layer theory. The Blasius equation with injection boundary condition $[f''' + ff'' = 0, f'(\infty) = 1, f(0) = -C, f'(0) = 0]$ was solved numerically with the result that as $C \to C_0 = 0{\cdot}87574...$, the wall shear $f''(0) \to 0$. Direct numerical calculation becomes increasingly more difficult, because the limiting condition of $f''(0) = 0$ implies that all the derivatives of $f(\eta)$ near the wall vanish.

As the injection rate approaches the critical value, the region of strong shear moves away from the wall causing a dramatic thickening of the boundary layer. In fact, most of the boundary layer is composed of a region of injected fluid in which the shear is very small. Above this is a relatively thin strong shear layer, which in the limit $C \to C_0$ becomes identical to the free mixing layer described by Lock (1951). The foregoing asymptotic properties of this famous numerical calculation have been described analytically by Kassoy (1970).

Physically, the blow-off phenomenon described in terms of boundary-layer theory is directly attributable to the mass addition effect (in distinction to the distributed momentum), because there is neither transverse inertia nor viscous effect associated with the fluid in classical boundary-layer theory (changes in the transverse velocity are determined strictly from mass conservation). The injected fluid simply fills the region near the wall, pushing the shear layer away. The characteristic length of the latter ultimately becomes smaller than the thickness of the former, and hence the viscous effect near the wall vanishes.

In the case of injection values larger than C_0 it is clear that a pressure force must be available to assist the viscous effect in turning the injected fluid. Pretsch (1944), and subsequently Acrivos (1962), Watson (1966), Aroesty & Cole (1967), and Kubota & Fernandez (1968) developed solutions to the Falkner–Skan equations ($f''' + ff'' + \beta(1-f'^2) = 0, f'(\infty) = 1, f(0) = -C, f'(0) = 0$), or the compressible analogue for large values of C, and $\beta = O(1)$; so-called hard blowing. The calculations in the latter references are based on a matched asymptotic expansion scheme for the limit $C \to \infty$. An inner region of small shear near the wall is described basically in terms of the reduced form of the momentum equation, $ff'' + \beta(1-f'^2) = 0$ (or its compressible analogue), which satisfies only the boundary conditions at the wall. In this inviscid rotational flow, the applied positive pressure force alone represented by β is responsible for turning the flow almost entirely toward the external flow direction. In fact, for the incompressible problem it was shown by Pretsch (1944) that the inner solution actually satisfies the external boundary condition. In general, however, an outer shear layer solution is necessary to provide a transition from the inner solution to the prescribed external boundary condition. In this outer layer both viscous and pressure forces combine to complete the required turning effect. The definitive analysis of Kubota & Fernandez (1968) is based on the use of the modified von Mises transformation, wherein the solution is developed in the form of $f' = g(f)$. However, several salient features of the flow are most usefully described by a solution in the original similarity variables $f = f(\eta)$. In appendix A of the current work the problem considered by Kubota & Fernandez is rederived briefly directly from the original equations. The technique which enables one to perform this calculation involves the use of purely translational asymptotic transformations described by Kassoy (1970). (Such transformations, it should be pointed out, are simply extensions of Prandtl's transposition theorem.) The results are useful for describing the thickness and location of the inner and shear layers, respectively, in an explicit fashion.

Insofar as similarity analysis is concerned, there remains the problem of injection rates of finite size but larger than the critical value; $C = O(1) > C_0$. One may infer from the previous discussion that a favourable pressure gradient, compatible with the assumption of small wall shear, must be available to help turn the flow. Thus, the present paper is concerned with the compressible similar laminar boundary-layer equations with finite values of injection larger than the critical value, when the wall shear is small. The solution is constructed in terms of an asymptotic analysis based on the small parameter $f''(0) = \epsilon$, wherein the relevant pressure gradient is developed. Alternatively, one could prescribe a small value of β as the parameter, and seek an expression for the wall shear.

The mathematical description of the flow involves an inviscid rotational inner layer, above which lies a relatively thinner viscous shear layer. In the former region, pressure forces alone are responsible for turning the injected fluid. The solution here (inner) is valid only to that distance from the wall where $f = -C_0$. Beyond that distance, in the outer region, it is found that Lock's mixing-layer solution provides the basic description. The relationship between the two regions is described in terms of the aforementioned translational transformation, the specific form of which is determined in the course of analysis. The results of the calculation provide a transitional solution valid between the classical blow-off solution exemplified by Kassoy (1970), in which the fundamental describing equations are basically viscous in nature, and the hard-blowing calculation, in which the inviscid rotational layer plays a major role. Thus, the present work describes the manner in which the new layer develops as a function of the injection rate C. The results also confirm Nickel's (1958) assertion that separation ($f''(0) \equiv 0$) cannot occur in any accelerated flow of the type considered here.

Perhaps the most interesting result to be obtained from the analysis is the overall boundary-layer characteristic thickness, which is found to be of $O([\epsilon\, Re]^{-\frac{1}{2}})$, where ϵ is the order of the wall shear. Hence, for a given large Reynolds number, a sufficiently small wall shear implies that the characteristic dimension is much larger than that of classical boundary-layer theory, $O(Re^{-\frac{1}{2}})$. It follows that the analysis, based as it is on classical boundary-layer theory ($Re \to \infty$), is not uniformly valid in the limit $\epsilon \to 0$. This difficulty, which arises as a result of the multiple limit nature of the problem, can be resolved by constructing a new distinguished limit of the Navier–Stokes equations based on $Re \to \infty$ alone. The development is described in appendix B for the special case of incompressible uniform flow past a flat plate from which fluid is being injected with an arbitrary distribution. A solution is presented for the case $v_\omega = C/(2x)^{\frac{1}{2}}$, where $C > C_0$.

2. Mathematical system

The formal describing system for compressible similar first-order laminar boundary-layer flow with mass addition may be written in the form,

$$\left.\begin{aligned} f''' + ff'' + \beta(g - f'^2) = \mathscr{L}(f) = 0,\\ g'' + fg' = \mathscr{M}(g) = 0,\\ f = -C, \quad f' = 0, \quad g = g_\omega, \quad \text{at} \quad \eta = 0,\\ f' = 1, \quad g = 1, \quad \text{at} \quad \eta \to \infty, \end{aligned}\right\} \qquad (1\,a\text{–}d)$$

where the dependent variables $f(\eta)$ the stream function, $g(\eta)$ the enthalpy, and the independent similarity variable η have the usual definitions (Moore 1964). In order to consider the mass-addition problem when the wall shear is small, it is convenient (although it would appear arbitrary at this juncture) to divide the system in (1) into two parts, by means of the similarity form of Prandtl's transposition theorem. The procedure outlined in §2 of Kassoy (1970) may be used to develop first an 'inner' system, in which the independent variable is η. This system is to satisfy only the wall boundary conditions in (1c), and the additional condition of small wall shear. Thus,

$$\left.\begin{aligned} \mathscr{L}(f[\eta]) &= 0, \\ \mathscr{M}(g[\eta]) &= 0, \\ f = -C, \quad f' = 0, \quad f'' = \epsilon, \quad g &= g_\omega \quad \text{at} \quad \eta = 0, \end{aligned}\right\} \qquad (2a\text{–}c)$$

where $\epsilon \ll 1$, and $C = O(1) > C_0$. The pressure gradient parameter β, ordinarily treated as a prescribed quantity, will be considered as an eigenvalue subject to compatability with the additional small wall shear condition. An estimate of the asymptotic behaviour of β may be inferred from the notion that near the wall the pressure force is the primary causative agent in turning the injected fluid which results in a shear stress. Hence, it seems likely that in the limit $\epsilon \to 0$, $\beta(\epsilon) \to 0$ at some asymptotic rate. It should be noted that this result is in contradistinction to that in Kubota & Fernandez (1968), wherein $\beta = O(1)$, even though the wall shear is small. The difference is attributable to the extreme magnitude of the injection rate in the hard-blowing problem implied by the limit $C \to \infty$.

If indeed $\beta(\epsilon)$ is an asymptotically small function of ϵ, then when the limit process, $\epsilon \to 0$ (η fixed), is applied to the 'inner' system in (2), the pressure gradient term is absent from the reduced system. However, if values of $C > C_0$ are to be considered, the pressure effect is absolutely essential. To this end, the application of the stretching transformation,

$$z = \epsilon^{\frac{1}{2}}\eta, \quad F(z) = f(\eta), \quad G(z) = g(\eta), \qquad (3)$$

in (2) leads to the system,

$$\left.\begin{aligned} \epsilon^{\frac{1}{2}}F''' + FF'' + (\beta/\epsilon)(G - \epsilon F'^2) &= 0, \\ \epsilon^{\frac{1}{2}}G'' + FG' &= 0, \\ F = -C, \quad F' = 0, \quad F'' = 1, \quad G &= g_\omega \quad \text{at} \quad z = 0, \end{aligned}\right\} \qquad (4a\text{–}c)$$

which is a distinguished limit that provides a relevant description of the wall region. According to (3), this wall region has a large characteristic length in η of $O(\epsilon^{-\frac{1}{2}})$. Equation (4$a$) implies that the viscous effect $\epsilon^{\frac{1}{2}}F'''$, is smaller than the largest inertial contribution. Furthermore, the required balance of inertia and pressure forces leads to the conclusion that $\beta = O(\epsilon)$, verifying the earlier estimate.

The outer system, derived by applying the similarity form of Prandtl's transposition theorem,

$$s = \eta - \lambda(\epsilon), \qquad (5)$$

to (1), is

$$\left.\begin{aligned} \mathscr{L}(f[s]) &= 0, \\ \mathscr{M}(g[s]) &= 0, \\ f = 0, \quad \text{at} \quad s &= 0, \\ f' = g = 1, \quad \text{for} \quad s &\to \infty. \end{aligned}\right\} \tag{6a–d}$$

Here the new condition in (6c) is used to indicate that $s = 0$ represents the dividing streamline. The transformation in (5), wherein $\lambda(\epsilon)$ is found to be a large positive asymptotic function of ϵ, is purely translational in nature. Since z is the more appropriate inner variable, it is useful to combine (3) and (5) in the form

$$s = (z - \epsilon^{\frac{1}{2}}\lambda(\epsilon))/\epsilon^{\frac{1}{2}}. \tag{7}$$

The form of (7) may be interpreted to mean that the outer layer, with a characteristic extent of $O(\epsilon^{\frac{1}{2}})$, lies above a relatively thicker inner layer of $O(1)$ in the z plane, or $O(\epsilon^{-\frac{1}{2}})$ in the η plane. Application of the limit process, $\epsilon \to 0$ (z fixed), implies that the z plane is reached from the s plane by letting $s \to -\infty$. Similarly, the use of the limit process, $\epsilon \to 0$ (s fixed), indicates that the s plane is reached from the z plane if $z \to \epsilon^{\frac{1}{2}}\lambda(\epsilon)$. These limit-process concepts are useful for constructing the matching conditions between the wall region solution and the outer region, which replace the missing boundary conditions in (4) and (6).

Formal solutions to the problem defined by (4)–(7) are to be constructed in terms of matched asymptotic expansions. Those for the wall region of the general form

$$Q(z;\epsilon) \sim \sum_{n=0} \mu_n(\epsilon)\, Q_n(z), \quad \lim_{\epsilon\to 0} \mu_{n+1}/\mu_n = 0, \tag{8a}$$

are defined by the limit process, $\epsilon \to 0$ (z fixed), whereas the outer expansions,

$$q(s;\epsilon) \sim \sum_{n=0} \gamma_n(\epsilon)\, q_n(s), \quad \lim_{\epsilon\to 0} \gamma_{n+1}/\gamma_n = 0, \tag{8b}$$

are defined by $\epsilon \to 0$ (s fixed). The asymptotic sequences must be determined. It is also necessary to define asymptotic expansions for the unknown parameters,

$$\left.\begin{aligned} \beta &= \epsilon\Big(K_0 + \sum_{n=1} \mu_n(\epsilon)\, K_n\Big), \\ \lambda &= \sum_{n=0} \alpha_n(\epsilon)\, k_n, \quad \lim_{\epsilon\to 0} \alpha_{n+1}/\alpha_n = 0, \end{aligned}\right\} \tag{9a, b}$$

where K_n and k_n are constants that must be determined. Finally, the matching procedure will be carried out in terms of

$$\lim_{\epsilon\to 0} [Q(z[s;\epsilon];\epsilon) \sim q(s;\epsilon)]. \tag{10}$$

3. Asymptotic analysis

The wall layer

When the expansions in (8a), (9a) are used in (4), and terms of similar order are gathered, it is found that $\mu_n = \epsilon^{\frac{1}{2}n}$, $n = 0, 1, 2, \ldots$. The resulting sequence of systems to second order is:

$$\left.\begin{aligned} F_0 F_0'' + K_0 G_0 = 0; \quad F_0(0) = -C, \quad F_0'(0) = 0, \quad F_0''(0) = 1; \\ F_0 G_0' = 0, \quad G_0(0) = g_\omega; \end{aligned}\right\} \tag{11a,b}$$

$$\left.\begin{aligned} F_0F_1''+F_0''F_1+F_0'''+K_0G_1+K_1G_0 = 0, \quad F_1(0) = F_1'(0) = F_1''(0) = 0; \\ F_0G_1'+G_0'F_1+G_0'' = 0, \quad G_1(0) = 0; \end{aligned}\right\} \tag{12a, b}$$

$$\left.\begin{aligned} F_0F_2''+F_0''F_2+F_1F_1''+F_1'''+K_0(G_2-F_0'^2)+K_1G_1+K_2G_0 = 0; \\ F_2(0) = F_2'(0) = F_2''(0) = 0, \\ F_0G_2'+G_0'F_2+F_1G_1'+G_1'' = 0; \quad G_2(0) = 0. \end{aligned}\right\} \tag{13a, b}$$

Each of these sets of equations, third-order in derivatives ω/r to z, and containing an eigenvalue K_n, satisfy four boundary conditions implying completely defined systems.

Wall-layer solutions

The system in (11) describes a basically inviscid rotational flow, whose dynamics are determined by a balance of pressure and inertia terms. So long as $F_0 \neq 0$, (11*b*) implies that $G_0' = 0$, or $G_0 = g_\omega$. Substitution of this result into (11*a*), and use of the wall-shear boundary condition, leads directly to $K_0 = C/g_\omega$. Finally, the closed form implicit solution is

$$z = (\pi C/2)^{\frac{1}{2}} \operatorname{erf}(\ln^{\frac{1}{2}}[C/-F_0]). \tag{14a}$$

It is also useful to note that

$$F_0' = (2C\ln[C/-F_0])^{\frac{1}{2}}, \tag{14b}$$

$$F_0'' = -C/F_0. \tag{14c}$$

These lowest-order results may now be used in (12). It follows that $G_1 = 0, K_1 = 0$, and that the reduced form of (12*a*) is

$$F_1'' - \frac{CF_1}{F_0^2} = -\frac{CF_0'}{F_0^3}; \quad F_1(0) = F_1'(0) = 0.$$

The particular solution to this equation can be constructed by use of variation of parameters, once it is noticed that the homogeneous solutions are $F_{11} = F_0'$, and $F_{12} = zF_0' - F_0$. Application of the boundary conditions leads to a completely defined result for F_1 in quadrature form:

$$F_1 = \frac{F_0'}{2C}\left[\int_0^{F_0'/C^{\frac{1}{2}}} e^{-\tau^2/2}\,d\tau \int_0^{\tau} e^{x^2}\,dx - 1\right] - \frac{F_0}{2C^{\frac{3}{2}}}\int_0^{F_0'/C^{\frac{1}{2}}} e^{\tau^2}\,d\tau. \tag{15}$$

Finally, the results in (14) and (15) may be utilized in (13) with the ultimate result that $G_2 = 0$, $K_2 = -1/g_\omega C^2$ and

$$F_2'' - \frac{C}{F_0^2}F_2 = -\frac{F_1F_1''}{F_0} - \frac{F_1'''}{F_0} + \frac{C}{g_\omega}\frac{F_0'^2}{F_0} + \frac{1}{C^2F_0},$$

the solution of which could also be developed in quadrature form. It may be surmised from these results that the wall layer is a constant enthalpy field to all small algebraic orders in ϵ. Even for the relatively small injection rates considered here, there is absolutely no heat transfer to the wall.

To second order, the pressure gradient eigenvalue has the form

$$\beta \sim \epsilon C/g_\omega - \epsilon^2/g_\omega C^2 + O(\mu_3). \tag{16}$$

If the small parameter in the problem is thought of as β rather than ϵ, the asymptotic expansion in (16) may be inverted to show that

$$f''(0) = \epsilon \sim g_\omega \beta / C + g_\omega^2 \beta^2 / C^5 + \ldots, \tag{17}$$

the leading term of which is identical in functional form to the result of Kubota & Fernandez (1968). It appears that, quite generally, cold walls, small pressure gradients, or large injection rates will lead to incipient blow-off phenomena, and vanishing heat transfer rates.

The solutions for F and G, containing no undetermined constants, are fully specified. Hence, their behaviour must be examined carefully to determine where the approximations, inherent in the description of the wall layer, fail. Except for the special case of $g_\omega = 1$, the form $G = g_\omega$ will not satisfy the external boundary condition, implying the existence of a thermal transition layer somewhere in the field. The nature and location of this layer will be more fully discernible after an investigation of the stream-function behaviour. It may be observed, from (14) and (15), that F increases from the wall value $-C$ with distance from the wall. An examination of (11a) will show that a singularity occurs when $F_0 \to 0^-$; according to (14a), the latter happens when $z \to (\pi C/2)^{\frac{1}{2}}$. In fact, a comparison of the neglected viscous term in (4a), $\epsilon^{\frac{1}{2}} F_0'''$ with the inertia term $F_0 F_0''$ shows that they are of the same order when $F_0 = O(\epsilon^{\frac{1}{2}} \ln^{\frac{1}{2}} 1/\epsilon)$ or when $F_0' = O(\ln^{\frac{1}{2}} 1/\epsilon)$. This result indicates that the wall region solution can in no way satisfy the external boundary condition, which in the z plane can be written as $F_0'(z \to \infty) = \epsilon^{-\frac{1}{2}}$. This value is much larger than the value of F_0' in the region where the wall-layer solution fails.

A further elucidation of this matter may be obtained by briefly examining the outer momentum equation in (6a) in the limit, $\epsilon \to 0$. Recalling that $\beta = O(\epsilon)$, the limiting form is the Blasius equation. Since the wall shear is small, it is reasonable to expect the asymptotic form of the outer solution (for $s \to -\infty$) to indicate a vanishing shear. Hence, it appears likely that the basic outer stream-function solution is identically Lock's mixing layer, where $f(s \to -\infty) \sim -C_0$. Then the transformation $f(s) = F(z)$ implies that the wall region solution should be truncated at the value of z, $z^* = (\pi C/2)^{\frac{1}{2}} \operatorname{erf}(\ln^{\frac{1}{2}} C/C_0)$, where $F = -C_0$, rather than continuing toward $z = (\pi C/2)^{\frac{1}{2}}$, where $F \to 0^-$. If this conjecture is correct (and it will be shown to be so), then, over the relatively large distance $\eta^* = z^*/\epsilon^{\frac{1}{2}}$, the weak pressure gradient has managed to turn the flow very gradually, so that to lowest order $f(\eta^*) = -C_0$, and $f'(\eta^*) = \epsilon^{\frac{1}{2}}(2C \ln C/C_0)^{\frac{1}{2}}$. The latter result implies that the x-wise velocity $u = f'(\eta)$ is still quite small. The remaining transition to the external flow occurs in the outer shear layer, which must be considered in detail.

An initial estimate of the magnitude of $\lambda(\epsilon)$, the translation parameter in (5) and (9b), can be ascertained from the fact that the wall region extends to $\eta^* = z^*/\epsilon^{\frac{1}{2}}$. This implies that the outer layer is translated away from the wall by a distance approximately $O(\epsilon^{-\frac{1}{2}})$, or that, as a first approximation,

$$\lambda(\epsilon) \sim k_0 \alpha_0(\epsilon) = z^*/\epsilon^{\frac{1}{2}}.$$

It follows from (7) that

$$s = (z - z^*)/\epsilon^{\frac{1}{2}} - \sum_{n=1} \alpha_n(\epsilon)\, k_n. \tag{18}$$

It should be noted that, in terms of the z plane, the origin of the outer layer $s = 0$ is located at

$$\hat{z} = z^* + \epsilon^{\frac{1}{2}} \sum_{n=1} \alpha_n(\epsilon)\, k_n,$$

which, for physical reasons if no other, must be slightly larger than z^*. Hence, at least $k_1 > 0$.

The form of the wall-layer solution to be used in the matching condition can now be constructed by interpreting (10) to mean that

$$f(s \to -\infty) \sim F(z \to z^*) = F(z^*) + \sum_{n=1}^{\infty} \frac{d^n F(z^*)}{dz^n} \frac{(z-z^*)^n}{n!}. \tag{19a}$$

When (14), (15) and (18) are substituted into (19a), the matching form follows:

$$\begin{aligned} f(s \to -\infty) \sim\; & -C_0 + \epsilon^{\frac{1}{2}}([2C \ln C/C_0]^{\frac{1}{2}} [s + \alpha_1(\epsilon) k_1 + \alpha_2(\epsilon) k_2 + \ldots] + F_1(z^*)) \\ & + \epsilon(C/2C_0[s + \alpha_1(\epsilon) k_1 + \alpha_2(\epsilon) k_2 + \ldots]^2 \\ & + F_1'(z^*) [s + \alpha_1(\epsilon) k_1 + \alpha_2(\epsilon) k_2 + \ldots] + F_2(z^*)) + O(\epsilon^{\frac{3}{2}}), \end{aligned} \tag{19b}$$

where the α_n, and k_n remain to be found. The values of $F_1(z^*)$, $F_1'(z^*)$ and $F_2(z^*)$ are obtained by integration of the appropriate equations to $z = z^*$. The analogous enthalpy matching form is $g(s \to -\infty) = g_\omega$. There are no higher-order corrections.

Outer layer

The outer system may be written explicitly by combining (6) and (16) into the form

$$\left.\begin{aligned} f''' + ff'' + (\epsilon C/g_\omega - \epsilon^2/g_\omega C^2 + \ldots)(g - f'^2) = 0, \\ g'' + fg' = 0, \\ f(s = 0) = 0, \quad f'(s \to \infty) = g(s \to \infty) = 1. \end{aligned}\right\} \tag{20a–c}$$

Substitution of the relevant asymptotic expansions defined in (8b), and the gathering of like-order terms, leads to a hierarchy of systems which describe the shear layer. The lowest-order system is:

$$f_0''' + f_0 f_0'' = 0; \quad f_0'(\infty) = 1, \quad f_0(0) = 0; \tag{21a}$$

$$g_0'' + f_0 g_0' = 0; \quad g_0(\infty) = 1; \tag{21b}$$

for this system, the matching conditions in (19b) imply that

$$f_0(s \to -\infty) = -C_0, \quad g_0(s \to -\infty) = g_\omega, \tag{21c}$$

where the approach to the asymptotic value is exponentially fast. The stream function $f_0(s)$, identical to that of Lock's mixing layer, and the enthalpy distribution g_0, must be developed *via* numerical computations, the form of which have been described by Lock (1951), Kassoy (1970) and Libby & Kassoy (1970).

The structure of the matching condition in (19) implies that the first correction in the outer layer must be $O(\epsilon^{\frac{1}{2}})$. Hence, if $\nu_1 = \epsilon^{\frac{1}{2}}$ (and it should be noted that the largest pressure effect is $O(\epsilon)$), then the first-order equations and boundary conditions are

$$\left.\begin{aligned} f_1''' + f_0 f_1'' + f_0'' f_1 = 0, \quad f_1'(\infty) = f_1(0) = 0, \\ g_1'' + f_0 g_1' + f_1 g_0' = 0, \quad g_1(\infty) = 0. \end{aligned}\right\} \tag{22a, b}$$

The solution to the first-order stream-function equation and boundary conditions (discussed by Stewartson 1964; Kassoy 1970) is

$$f_1(s) = C_1\left[f_0'(s)\int_0^s \frac{f_0''(\tau f_0'+f_0)\,d\tau}{(2f_0'^2+f_0''')^2} + (sf_0'+f_0)\int_s^\infty \frac{f_0' f_0''\,d\tau}{(2f_0'^2+f_0''')^2}\right], \tag{23}$$

where C_1 is an integration constant, and τ a dummy integration variable. The asymptotic form of f_1,

$$f_1(s\to-\infty) \sim C_1\left[-\frac{1}{C_0^2}(s+\hat{s}+O(s^2 e^{C_0 s}))\right], \tag{24}$$

where $\hat{s} = 2\cdot337$ must match with the $O(\epsilon^{\frac{1}{2}})$ term in (19*b*). It follows from comparison of the $O(s)$ terms that

$$C_1 = -C_0^2(2C\ln(C/C_0))^{\frac{1}{2}}, \tag{25a}$$

and, from the $O(1)$ terms, that

$$\alpha_1(\epsilon) = 1, \quad k_1 = \hat{s} - F_1(z^*)/(2C\ln(C/C_0))^{\frac{1}{2}}. \tag{25b, c}$$

The latter formulae are necessary, because, in general, $\hat{s} \neq F_1(z^*)/(2C\ln(C/C_0))^{\frac{1}{2}}$. It should be noted that, if the matching condition were written more formally in terms of an intermediate variable, the exponential terms in (24) would become transcendentally small in the parameter.

The first-order enthalpy system is completed by using the matching form to show that $g_1(s\to-\infty) = 0$, implying the necessity of exponential decay which is compatible with the equations. Here again the full solutions may be obtained by numerical integration.

One may now proceed to the second correction to the outer flow, in which the weak pressure gradient finally influences the mathematical structure. Both the $O(\epsilon)$ term in the matching condition, and that in the basic outer equation imply that $\nu_2 = \epsilon$, so that the second-order equations and boundary conditions are

$$\left.\begin{aligned} f_2'''+f_0f_2''+f_0''f_2 &= -f_1f_1''-(C/g_\omega)(g_0-f_0'^2), \quad f_2'(\infty)=f_2(0)=0,\\ g_2''+f_0g_2' &= -f_1g_1'-f_2g_0', \quad g_2(\infty)=0. \end{aligned}\right\} \tag{26a, b}$$

The complete solution to (26*a*) could be constructed in quadrature form from the known homogeneous solutions and variation of parameters. In symbolic form,

$$f_2 = C_2\left[f_0'(s)\int_0^s \frac{f_0''(\tau f_0'+f_0)\,d\tau}{(2f_0'^2+f_0''')^2} + (sf_0'+f_0)\int_s^\infty \frac{f_0 f_0''\,d\tau}{(2f_0'^2+f_0''')^2}\right] - f_0'(s)\frac{f_{2p}(0)}{f_0'(0)} + f_{2p}(s), \tag{27}$$

where $f_{2p}(s)$ is the particular solution. In the asymptotic sense, $s\to-\infty$ the term in brackets is $O(s)$, while $f_0' = O(e^{C_0 s})$. According to (19*b*),

$$f_2(s\to-\infty) = (C/2C_0)\,s^2+O(s)$$

implying that the leading asymptotic term in $f_{2p}(s)$ must be $O(s^2)$. This requirement can be verified by using the asymptotic form of (26*a*),

$$f_2''' - C_0f_2'' + O(e^{C_0 s}f_2) = -C + O(s^3 e^{C_0 s}),$$

from which it follows that $f_2(s\to-\infty) \sim (C/2C_0)\,s^2 + O(s)$. One could, of course, carry out the full second-order matching by calculating the fuller asymptotic form of (27). A procedure, similar to that used in the first-order matching, implies that $\alpha_2 = \epsilon^{\frac{1}{2}}$, and would give numerical values for C_2 and k_2. The details, of immense algebraic complexity, add little to the fundamentals of the problem, and will not be discussed.

4. Interpretation

The wall-layer solution describes a basically rotational inviscid flow, in which the source of vorticity is the wall (the condition $f''(0) = \epsilon$), as has been discussed by Lighthill in Rosenhead (1963, ch. 5). The relatively weak pressure gradient $\beta = O(\epsilon)$ (see (16)) acts over a relatively large distance $\eta = O(\epsilon^{-\frac{1}{2}})$, to turn the injected fluid slightly. The solution in the wall region, where the enthalpy is constant, is valid to $\eta^* = z^*/\epsilon^{\frac{1}{2}}$, at which point $f(\eta^*) \sim -C_0 + O(\epsilon^{\frac{1}{2}})$, and $f'(\eta^*) = O(\epsilon^{\frac{1}{2}})$. An examination of the functional relationship between z^* and C leads to the conclusion that, for a given ϵ, η^* increases (decreases) with increasing (decreasing) values of C. It is to be expected that increasing the blowing rate will thicken the inviscid layer. On the other hand, should $C \to C_0$ for a given ϵ, it may be observed that $z^* \to 0$, causing the wall layer to vanish. This result implies that the analysis fails because it is predicated upon the assumption of a thick inner layer. In fact, it is possible to show that, so long as $\ln C/C_0 \gg \epsilon$, the solution is valid.

On the other hand, one can also consider the hard blowing limit $C \to \infty$, for which it is found that $z^* \sim (\pi C/2)^{\frac{1}{2}}(1 + O(C^{-\frac{1}{2}}\ln^{-\frac{1}{2}} C))$, or, in the η plane, that $\eta^* \sim (\pi C/2\epsilon)^{\frac{1}{2}}$. If the ϵ dependence is replaced by β, using (16), it follows that $\eta^* \sim (\pi C^2/2\beta g_\omega)^{\frac{1}{2}} = O(C)$ with respect to the limit $C \to \infty$. This result implies that for a fixed βg_ω, the extent of the wall region is linearly proportional to C, which is identical to the hard-blowing analysis, as may be seen in appendix A. A further comparison of the present results with the hard-blowing calculation follows from considering the latter for $\beta \ll 1$ and the former for $C \to \infty$. The first integral of (A 3*a*), and the relevant co-ordinate transformations, imply that $f'(\eta) \sim [2\beta g_\omega \ln C/|f|]^{\frac{1}{2}}$, which is identical to the result found from (14*b*) with the appropriate transformations, the use of (16), and the limit $C \to \infty$. Hence, the present solution gives a natural transition to the hard-blowing result when β is assumed to be small.

Beyond z^* lies the thinner shear layer, in which viscous effects are predominantly responsible for the remaining transition to the external flow, and in which all the enthalpy variation occurs. Lock's mixing layer describes the basic stream function with pressure gradient effects entering as a second-order correction. Equations (5), (7), (9), and the results of the analysis, indicate that the dividing streamline defined by $f(s = 0) = 0$ is located in the η plane at $\hat{\eta} = (z^*/\epsilon^{\frac{1}{2}}) + k_1 = \eta^* + k_1$. Hence, the 'shifting' constant k_1 (see (25*c*)) may be interpreted as the distance in the η plane between the outer edge of the wall region and the origin of the shear layer. By using the functional relation for z^*, and (16), one may write

$$\hat{\eta} = C(\pi/2\beta g_\omega)^{\frac{1}{2}} \operatorname{erf}(\ln^{\frac{1}{2}}(C/C_0)) + k_1,$$

in which the error function is a weak function of C. Hence, for a specified small β (rather than ϵ) the location of the dividing streamline is almost linearly proportional to C.

5. Numerical results

Some verification of the analytical procedure may be obtained by comparing the results with those of numerical computation. To this end, the system in (1) was numerically integrated for several combinations of β, g_ω, and $C > C_0$ such that $\epsilon \sim \beta g_\omega / C \ll 1$. Table 1 contains the wall-shear values found from (17), and those computed. The agreement is excellent in all cases except the first and fifth. Here the accuracy of numerical computation is to be doubted, because

C	β	g_ω	$f''(0)$; equation (17)	$f''(0)$; num.	$\hat{\eta}$	$\hat{\eta}$; num.
0·9	0·01	0·1	0·0011	0·00176	7·93	8·15
2·0	0·2	1·0	0·1013	0·1009	5·68	5·71
2·0	0·5	0·5	0·1280	0·1251	5·47	5·48
2·0	0·1	1·0	0·0501	0·0508	7·82	7·84
2·0	0·1	0·1	0·0051	0·0039	21·31	22·05
4·0	0·3	2·0	0·1501	0·1501	6·99	7·05
4·0	0·5	0·5	0·0630	0·0625	10·98	11·12

TABLE 1. Comparison of analytically and numerically obtained values of wall shear and dividing streamline location.

the combination of parameters imply that the flow field, being extremely close to blow-off, contains vanishingly small derivatives near the wall, which the computer cannot handle adequately. A second comparison involves the location of the dividing streamline in the η plane. Here again rather nice agreement is obtained.

6. Conclusions

The foregoing analysis indicates that incipient blow-off phenomena can occur for flows with specified small favourable pressure gradients and modest values of injection. Of particular practical interest is the conclusion that the heat transfer vanishes completely under the prescribed circumstances.

The problem has been modelled in terms of an inviscid rotational wall layer, and a thin viscous layer. It must be emphasized that the pressure effect, of secondary importance in the shear layer, is the essential feature of the wall layer. Without it, solutions are limited to $C < C_0$. In a related problem, Inger (1969) has noted that "...the presence of even a small axial pressure gradient may have a profound effect on the basic mathematical structure of any solution one would seek to construct using matched inner-outer asymptotic expansions". It should also be realized that the viscous layer (although relatively thin) cannot be treated as a discontinuity in the present problem. In contrast to several massive blowing calculations based on a purely inviscid model with a slip streamline representing the collapsed shear layer (Pretsch 1944; Watson 1966; Cole & Aroesty 1967; Wallace & Kemp 1969), the present work features the thin mixing layer as a transition between the inviscid rotational layer and the external flow. Direct matching of the wall and external flow is not possible. In this regard, it is notable that in Cole & Aroesty (1967) it is found that the inviscid massive blowing model does not lead to complete solutions for certain types of power-law distributions of injection. The singularity arising in these cases precludes the possibility of a direct matching of the two inviscid layers. Thus, Cole & Aroesty (1967) state that a "calculation of a free shear layer requires further discussion". It would appear judicious to employ the present approach in those problems.

According to the transformation in (3), the wall region has a characteristic thickness in the η plane of $O(\epsilon^{-\frac{1}{2}})$. Then, from $\eta = y/(2x)^{\frac{1}{2}}$, where $y = y'Re^{\frac{1}{2}}/L$, y' is the physical transverse co-ordinate, and L is a characteristic longitudinal length, it follows that $y'/L = O(1/\epsilon^{\frac{1}{2}}Re^{\frac{1}{2}})$. For a sufficiently small wall shear, the classical boundary-layer approximation fails. The analogous result for the hard-blowing calculation in appendix A is $y'/L = O(1/\epsilon Re^{\frac{1}{2}})$, while for the blow-off problem of Kassoy (1970) the result is $y'/L = O(\ln(1/\epsilon)/Re^{\frac{1}{2}})$. For a given Reynolds number, the former has a characteristic dimension much larger than that of the present work, in contrast to the latter, which is relatively thinner. The failure of boundary-layer theory in all three cases for the limit $\epsilon \to 0$ implies that a new distinguished limit of the full describing equations is necessary to help provide a solution uniformly valid in the limit $Re \to \infty$, in which the wall shear is small. In appendix B an example of the relevant systems of equations for incompressible flow has been derived for boundary-layer type injection $v'/U_\infty = O(Re^{-\frac{1}{2}})$ and small wall shear. It is found that the inviscid rotational wall layer of $O(Re^{-\frac{1}{3}})$ is described by a set of equations similar to those found by Cole & Aroesty (1967) in the 'blow-hard' problem. Above and adjacent to the wall layer, there exists a conventional shear layer of thickness $O(Re^{-\frac{1}{2}})$, adjacent to the inner layer, in which the transition to the external inviscid irrotational flow occurs. A specific solution is presented for the case $v'/U_\infty = C/(2x)^{\frac{1}{2}}$ where $C > C_0 = 0{\cdot}87574\ldots$.

The author would like to express his gratitude to Paul Libby, Arthur Messiter, William Bush and George Inger, with whom he had numerous enlightening discussions. Thanks also go to T. M. Liu, and Nelson Kemp, who provided several numerical results. This work was partially supported by a Summer Research Initiation Faculty Fellowship granted by the University of Colorado, and by an N.S.F. grant GK-24689.

Appendix A

The system that describes compressible laminar boundary-layer flow with hard blowing is

$$\left.\begin{aligned} f''' + ff'' + \beta[g - f'^2] &= 0, \\ g'' + fg' &= 0, \\ f(0) = -C, \quad f'(0) = 0, \quad g(0) = g_\omega, \quad f'(\infty) &= g(\infty) = 1, \end{aligned}\right\} \qquad (\text{A } 1a\text{–}c)$$

where $C \gg 1$, and β is a prescribed parameter such that $0 < \beta < \frac{1}{2}$ for present purposes.

A solution is sought, valid in the limit $C \to \infty$. The η plane is not particularly well suited for developing the analysis. Rather, the stretching transformation $s = C^{-1}\eta$, $F = C^{-1}f$, $G = g$ is used to define a thick (in terms of η) inner wall region, in which the effects of viscosity are quite weak. Application of the preceding transformation to (A 1a–c) will result in the system,

$$\left.\begin{array}{c} C^{-2}F''' + FF'' + \beta[G - F'^2] = 0, \\ C^{-2}G'' + FG' = 0, \\ F = -1, \quad F' = 0, \quad G = g_\omega \quad \text{at} \quad z = 0. \end{array}\right\} \qquad (\text{A } 2a\text{–}c)$$

The lowest-order approximation, based on the limit process, $C^{-1} \to 0$ (s fixed),

$$\left.\begin{array}{c} F_0F_0'' + \beta[G_0 - F_0'^2] = 0, \quad F_0(0) = -1, \quad F_0'(0) = 0, \\ F_0G_0' = 0, \quad G_0(0) = g_\omega, \end{array}\right\} \qquad (\text{A } 3a\text{–}b)$$

describes an inviscid rotational flow. Three prescribed boundary conditions for this third-order system indicate that the solution is completely specified. From (A 3b) the temperature in the region is $G_0 = g_\omega$ implying zero heat transfer to lowest order. A first integral of the momentum equation, $F_0'^2 = g_\omega[1 - |F_0|^{2\beta}]$, and the resulting form of (A 3a), $F_0'' = \beta g_\omega |F_0|^{2\beta - 1}$, indicates that the inner solution is useful for $-1 \leqslant F_0 < 0$ because of the singularity at $F = 0$. The complete solution to (A 3a) in quadrature form,

$$s = g_\omega^{-\frac{1}{2}} \int_{-1}^{F_0} \frac{d\sigma}{[1 - |\sigma|^{2\beta}]^{\frac{1}{2}}}, \qquad (\text{A } 4)$$

can be used to define a value s_0 where $F_0(s_0) = 0$. Except for the isothermal case $g_\omega = 1$, the inner solution will not satisfy the external boundary condition as $s \to s_0$. Even for the special case there will be a discontinuity in the derivative. In general, a transitional shear layer must exist, centred around s_0.

In terms of the η plane, the point s_0 is located at $\eta_0 = Cs_0$. Hence, the shear-layer system may be derived formally in terms of a purely translational transformation defined by $r \equiv \eta - s_0 C = (s - s_0)C$. The formal equations and external boundary conditions are identical in form to (A 1a, b, c) with z replaced by r. The inner boundary condition is replaced by a matching condition applied at $r \to -\infty$. Formally, this condition is developed by calculating the asymptotic form of (A 4) for $s \to s_0$, and using the translational transformation and $F = C^{-1}f$. It follows that the lowest-order shear-layer system is

$$\left.\begin{array}{c} f_0''' + f_0 f_0'' + \beta[g - f_0'^2] = 0, \\ g_0'' + f_0 g_0' = 0, \\ f_0' = 1, \quad g_0 = 1, \quad \text{for} \quad r \to \infty, \\ f_0 = g_\omega^{\frac{1}{2}} r, \quad g = g_\omega, \quad \text{for} \quad r \to -\infty, \end{array}\right\} \qquad (\text{A } 5a\text{–}d)$$

when $g_\omega = 1$, the exact solution is trivial: $g_0 = 1, f_0 = r$. In other cases, numerical calculations are necessary. The overall results will be of course identical to those of Kubota & Fernandez (1968). It is to be noted that, for the case $g_\omega > 1$, the boundary condition in (A 5d) suggests that velocity over-shoot occurs. Higher-order solutions, analogous to those of Kubota & Fernandez, may be found if desired.

The lowest-order shear at the wall may be found directly from (A 3a), and the solution $G_0 = g_\omega$. Hence, $F_0''(0) = +\beta g_\omega$. It follows that the basic wall shear in the η plane is $f''(0) \sim \beta g_\omega / C$.

Appendix B

The results of classical boundary-layer theory for injection problems with small wall shear ($f''(0) = \epsilon$) contain a non-uniformity due to the multi-limit nature of the problem ($Re \to \infty$, $\epsilon \to 0$). In order to produce a uniformly valid solution for the limit $Re \to \infty$, it is necessary to reconsider the problem in terms of a theory which accounts directly for the drastic thickening of the injectant layer that has been shown to occur. To this end we construct an interaction problem for uniform incompressible, two-dimensional flow past a semi-infinite flat plate from which fluid is being injected at a classical boundary-layer rate $O(Re^{-\frac{1}{2}})$ under conditions which permit the wall shear to become very small. The layer of injected fluid, being thick compared to a classical boundary layer, will in effect induce a small perturbation on the uniform stream, with a resulting favourable pressure gradient, which prevents a zero wall shear condition from occurring.

The describing equations for the external flow (potential flow with a correction), the inner rotational but inviscid flow, and the boundary-layer-like shear layer are developed for a general injection distribution. A specific solution is presented for the case of the similarity distribution $v_\omega = c(2x)^{-\frac{1}{2}}$, associated with the Blasius equation in classical boundary-layer theory.

The formal mathematical description of the problem is given by the familiar equations and boundary conditions,

$$\left.\begin{aligned} \left\{\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - Re^{-1}\nabla^2\right\}\nabla^2\psi &= 0,\\ \psi_y(r \to \infty) &= 1,\\ \psi_y(x,0) = 0 \quad (x > 0)&,\\ \psi_x(x,0) = -B(x)\,Re^{-\frac{1}{2}} \quad (x>0)&, \end{aligned}\right\} \tag{B 1a–d}$$

where ∇^2 is the two-dimensional Laplacian operator, $r = x^2 + y^2$, Re is the Reynolds number and $B(x)$ represents the x-wise injection distribution. In the limit, $Re \to \infty$ (x, y fixed), the reduced form of (B 1a) is the Euler equation, the injection vanishes, and the system (B 1a, b, d) describes uniform flow past the flat plate. The displacement effect correction to this potential flow is found by assuming that

$$\psi(x, y; Re) \sim y + \sum_{n=1} \delta_n(Re)\,\psi_n(x,y), \tag{B 2}$$

where $Re^{-\frac{1}{2}} \ll \delta_1 \ll 1$, the lower limit suggested by the fact that the inner-most layers will have an effective dimension larger than the usual boundary-layer thickness. It follows that the stream-function correction is described by the system,

$$\frac{\partial}{\partial x}\nabla^2\psi_1 = 0, \quad \psi_1(r \to \infty) = 0, \tag{B 3}$$

and a matching condition with the outer edge of the shear layer which may be written formally as $\psi_{1_x}(x, y \to 0) = -\Phi(x)$, where $\Phi(x)$ is analogous to the slope of the 'effective displacement body' composed of the combination of the two inner layers, and must be found. The specified external flow is irrotational, implying that $\nabla^2\psi_1 = 0$. It follows that a general solution may be written in integral form (Van Dyke 1964). Insofar as finding the pressure correction imposed on the inner layers is concerned, an application of the linearized Bernoulli equation leads to the result that

$$p_1(x, 0) = -u_1(x, 0) = -\frac{1}{\pi}\int_0^\infty \frac{\Phi(\xi)\,d\xi}{x-\xi}, \quad x > 0 \tag{B 4}$$

which describes the pressure distribution on the fictitious displacement body surface. The function $\Phi(\xi)$ is found by constructing solutions in the inner layers.

The structure of the solutions considered in the main body of the present work suggests that, to describe the flow near the wall, one should seek a distinguished limit of the Navier–Stokes equation based on the limit $Re \to \infty$ which is basically inviscid but rotational in nature. This form will be obtained only if the transformation, $\overline{\psi} = Re^{\frac{1}{2}}\psi$, $\overline{y} = Re^{\frac{1}{3}}y$, is used in (B 1*a*) to show that

$$\left[\overline{\psi}_{\overline{y}}\frac{\partial}{\partial x} - \overline{\psi}_x\frac{\partial}{\partial \overline{y}} - Re^{-\frac{1}{6}}\frac{\partial^2}{\partial \overline{y}^2} - Re^{-\frac{5}{6}}\frac{\partial^2}{\partial x^2}\right][\overline{\psi}_{\overline{y}\overline{y}} + Re^{-\frac{2}{3}}\overline{\psi}_{xx}] = 0, \tag{B 5a}$$

subject to the boundary conditions, $\overline{\psi}_{\overline{y}}(x, 0) = 0$, $\overline{\psi}_x(x, 0) = -B(x)$, the reduced form of which is

$$\frac{\overline{D}}{Dt}\overline{\psi}_{0\overline{y}\overline{y}} = 0, \quad \overline{\psi}_{0\overline{y}}(x, 0) = 0, \quad \overline{\psi}_{0x}(x, 0) = -B(x), \tag{B 5b}$$

which implies conservation of vorticity on streamlines and a pressure distribution $\overline{p}_0 = \overline{p}_0(x)$. It is noted that the correction to the basic solution $\overline{\psi}_0$ is apparently $O(Re^{-\frac{1}{6}})$. Hence, (B 5*a*, *b*) describes the desired type of flow in a region whose characteristic dimension in $y = O(Re^{-\frac{1}{3}})$, thicker than the classical boundary-layer value, in which the shear is ordered by $\psi_{yy} = O(Re^{\frac{1}{6}})$, smaller than the boundary-layer value of order $Re^{\frac{1}{2}}$, and the x-wise velocity $u = O(Re^{-\frac{1}{6}})$.

The lateral extent of the wall layer, and consideration of its role as the effective displacement body, implies that the correction to the uniform flow is specified by $\delta_1(Re) = Re^{-\frac{1}{3}}$. The system is quite similar to that of Cole & Aroesty (1967). However, unlike their result, which was developed for so-called blow-hard problems, where $v = -\psi_x \gg O(Re^{-\frac{1}{2}})$, the present calculation involves classical injection rates; $v = O(Re^{-\frac{1}{2}})$.

The problem in (B 5) must be solved in terms of the unknown interaction pressure distribution $\overline{p}_0(x)$. In general, the solution will not match directly with the external potential flow described by (B 1) (although in special cases, such as those discussed by Cole 1967, *lowest-order* direct matching *is* possible). Rather, a classical free shear layer, whose characteristic dimension is $O(Re^{-\frac{1}{2}})$, must be interposed between the wall and external flows, in order to provide a transitional flow. The equations that describe this shear layer are derived from (B 1) by means of the transformation,

$$z = [y - g(x; Re)]\,Re^{\frac{1}{2}}, \quad \tilde{\psi} = Re^{\frac{1}{2}}\psi, \tag{B 6}$$

in which $y_D = g(x; Re)$, and $z = 0$ represents the equation describing the location of the dividing streamline $\psi = 0$ (and hence is representative of the effective displacement body). Consideration of the wall-layer configuration suggests that $g = O(Re^{-\frac{1}{3}})$, with a x dependence which must be determined. Substitution of (B 6) into (B 1a) will lead to

$$\left[\tilde{\psi}_z \frac{\partial}{\partial x} - \tilde{\psi}_x \frac{\partial}{\partial z} - (1 - g'^2)\frac{\partial^2}{\partial z^2} + Re^{-\frac{1}{2}}\left(2g'\frac{\partial^2}{\partial x \partial z} + g''\frac{\partial}{\partial z}\right)\right.$$
$$\left. - Re^{-1}\frac{\partial^2}{\partial x^2}\right][(1+g'^2)\tilde{\psi}_{zz} - Re^{-\frac{1}{2}}(2g'\tilde{\psi}_{xz} + g''\tilde{\psi}_z) + Re^{-1}\tilde{\psi}_{xx}] = 0,$$

the reduced form of which (for $g = O(Re^{-\frac{1}{3}})$) is the classical boundary-layer equation,

$$\left[\psi_{0z}\frac{\partial}{\partial x} - \psi_{0x}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2}\right]\tilde{\psi}_{0zz} = 0. \tag{B 7}$$

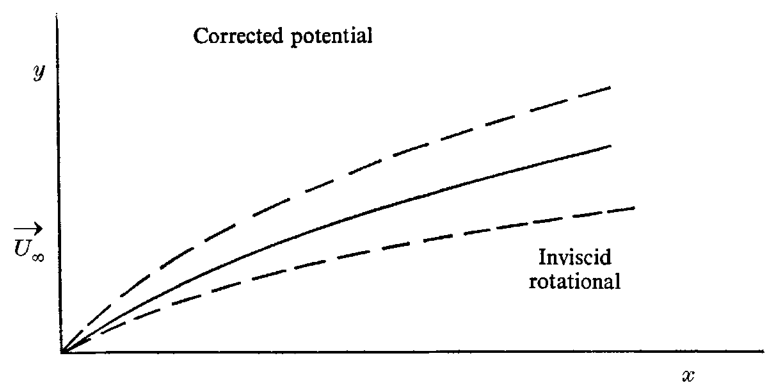

FIGURE 1. Flow structure: - - - - -, boundary of the viscous shear layer $O(Re^{-\frac{1}{2}})$; ——, dividing streamline $Y_D = g(x, Re)$; inviscid rotational layer $O(Re^{-\frac{1}{3}})$.

The usual boundary conditions are replaced by matching conditions at $z \to \pm\infty$, which may be inferred from the results of applying the limit process $Re \to \infty$ (y fixed) to the first of (B 6) for $y > y_D$ and $y < y_D$. The former has the form $\tilde{\psi}_0(x, z \to \infty) = z$, which is found by matching with the external flow. The latter will be discussed, with regard to a specific injection distribution, shortly. Finally, the condition for the zero streamline, $\tilde{\psi}(x, z = 0) = 0$, serves to define the function $g(x; Re)$. It should be noted that the shear layer described by (B 7) is not affected by the induced pressure gradient, which is a relatively higher-order effect.

To this point the systems describing the basic approximations in each of the three zones (figure 1) have been developed. It remains to consider specific injection distribution values, and ultimately find the pressure interaction term $\bar{p}_0 = \bar{p}_0(x)$ and $g(x; Re)$.

A similarity solution for $B(x) = C/(2x)^{\frac{1}{2}}$

A particularly simple solution for the systems in (B 1)–(B 7) can be constructed for the case of a classical similarity type injection distribution. $B(x) = C/(2x)^{\frac{1}{2}}$, where, as we shall find, $C > C_0 = 0{\cdot}87574$. The first integral of (B 5*b*) and the relevant boundary conditions,

$$\left.\begin{aligned}\bar{\Psi}_{0\bar{y}}\bar{\Psi}_{0x\bar{y}} - \bar{\Psi}_{0x}\bar{\Psi}_{0\bar{y}\bar{y}} &= -d\bar{p}_0/dx,\\ \bar{\Psi}_{0\bar{y}}(x,0) = 0, \quad \bar{\Psi}_0(x,0) &= -C(2x)^{\frac{1}{2}},\end{aligned}\right\} \qquad \text{(B 8}a, b\text{)}$$

describe the wall-layer flow. The pressure gradient remains to be found. This system can be satisfied by a similarity solution defined by

$$\bar{\Psi}_0 = (2x)^{\frac{1}{2}} f(\eta), \quad \eta = \bar{y}/(2x)^{\lambda/2}, \qquad \text{(B 9)}$$

where the constant $\lambda \neq 1$,† will be found in the process of analysis. Substitution of (B 9) into (B 8) leads ultimately to the similarity system,

$$\left.\begin{aligned}(1-\lambda)f'^2 - ff'' &= \beta,\\ f'(0) = 0, \quad f(0) &= -C,\end{aligned}\right\} \qquad \text{(B 10}a, b\text{)}$$

and a pressure distribution of the form,

$$\bar{p}_0(x) = \beta(2x)^{-(\lambda-1)}/2(\lambda-1), \qquad \text{(B 11)}$$

where β is a positive constant (to ensure a favourable pressure gradient), which must be evaluated along with λ. The first integral of (B 10*a*) has the form,

$$f'(\eta) = [\beta/(\lambda-1)]^{\frac{1}{2}}[(C/-f)^{2(\lambda-1)} - 1]^{\frac{1}{2}}, \qquad \text{(B 12)}$$

implying that for $f \to 0^-$ the x-wise velocity in the wall layer is singular, and that $\lambda > 1$. A second integral leads to the result,

$$\eta = -[(\lambda-1)/\beta]^{\frac{1}{2}} \int_C^{-f} [(C/\sigma)^{2(\lambda-1)} - 1]^{-\frac{1}{2}} d\sigma, \qquad \text{(B 13)}$$

which describes the increase in the value of the stream function from the most negative prescribed value at the wall with increasing η. The form of (B 12) implies clearly that the solution in (B 13) can be used for at most finite values of η, up to that for which $f \to 0^-$. As we shall see, however, the solution must be truncated at a smaller value of the independent variable, defined as η^*, which is determined essentially by matching with the shear-layer solution. The value of η^* defines a surface in the $x, \bar{y}$ plane with the equation, $\bar{y}^* = \eta^*(2x)^{\lambda/2}$, on which

$$\bar{\Psi}_0^* = (2x)^{\frac{1}{2}} f(\eta^*).$$

† If $\lambda = 1$, the similarity lines are parabolas. It would follow that no pressure gradient correction exists.

The first integral of the shear-layer equation in (B 7) can be written in the form

$$\tilde{\psi}_{0z}\tilde{\psi}_{0xz} - \tilde{\psi}_{0x}\tilde{\psi}_{0zz} = \tilde{\psi}_{0zzz}, \tag{B 14}$$

since the induced pressure gradient cannot affect the shear layer to lowest order. The boundary condition at the outer edge is $\tilde{\psi}_{0_z}(x, z \to \infty) = 1$. The matching form with the wall-layer solution is constructed by using (B 6) and the $\bar{\psi}$, $\bar{y}$ transformations defined above (B 5a), to show that $\tilde{\psi}_0(x, z \to -\infty) \sim (2x)^{\frac{1}{2}} f(\eta^*)$. Hence, the system of the shear layer admits the classical similarity solution defined by

$$\tilde{\psi}_0 = (2x)^{\frac{1}{2}} F(s), \quad s = z(2x)^{-\frac{1}{2}},$$

with the resulting similarity system,

$$\left.\begin{aligned} F''' + FF'' &= 0, \\ F'(s \to \infty) &= 1, \\ F(s = 0) &= 0, \\ F(s \to -\infty) &= f(\eta^*), \end{aligned}\right\} \tag{B 15a–d}$$

where the second boundary condition follows from the fact that $z = s = 0$ defines the location of the dividing streamline $\psi = 0$. This system is identical to Lock's mixing layer, from which it follows that $f(\eta^*) = -C_0 = -0{\cdot}87574\ldots$. Hence, in principle one can calculate the value of η^* from (B 13), where the upper limit on the integral is $-C_0$. The resulting η^*, which is smaller than that value of η for which $-f \to 0^-$, can be used to show that

$$y^* = \bar{y}^* Re^{-\frac{1}{3}} = \eta^*(2x)^{\lambda/2} Re^{-\frac{1}{3}}$$

is, for any given x, less than $y_D = g(x; Re)$ (the dividing streamline location), as shown in figure 1. However, in terms of the y plane the difference (of $O(1)$ in the $\bar{y}$ plane) must be small: $O(Re^{-\frac{1}{3}})$. Recalling in addition that

$$g(x; Re) = O(Re^{-\frac{1}{3}}),$$

the full form of the y, z transformation in (B 6) may be written generally as

$$z = [y - \eta^*(2x)^{\frac{1}{2}\lambda} Re^{-\frac{1}{3}} + \sum_{n=0} \alpha_n(Re)\, g_n(x)]\, Re^{\frac{1}{2}}, \tag{B 16}$$

where $\alpha_0 = o(Re^{-\frac{1}{3}})$, $\lim\,(\alpha_n + 1)/\alpha_n = 0$ for $n \geqslant 1$, and the sequences α_n, g_n have to be found in the process of analysis. The summation must be greater than zero, since $y_D > y^*$.

The interaction problem can be completed by constructing the higher-order matching between the external and shear-layer flows. The second part of (B 6) can be used to show that $\psi(x, y \to 0) \sim Re^{-\frac{1}{2}}\tilde{\psi}(x, z \to \infty)$, from which it follows that

$$\begin{aligned} y + Re^{-\frac{1}{3}}\psi_1(x, y \to 0) + \ldots &\sim Re^{-\frac{1}{2}}(2x)^{\frac{1}{2}} F(s \to \infty) \\ &\sim y - Re^{-\frac{1}{3}}\eta^*(2x)^{\frac{1}{2}\lambda} + o(Re^{-\frac{1}{3}}), \end{aligned} \tag{B 17}$$

which is constructed from (B 15b), $s = z(2x)^{-\frac{1}{2}}$ and (B 16). Matching requires that $\psi_1(x, y \to 0) \sim -\eta^*(2x)^{\frac{1}{2}\lambda}$, or that $v_1(x, y \to 0) = \phi_{1y}(x, y \to 0) = \eta^* \frac{1}{2}\lambda\,(2x)^{\frac{1}{2}(\lambda-2)}$, so that the system describing the first-order external flow correction, found from (B 3), is

$$\nabla^2\psi_1 = 0,$$
$$\psi_1(r\to\infty) = 0, \quad \psi_{1x}(x, y\to 0) = -\eta^*(\lambda/2)(2x)^{\frac{1}{2}(\lambda-2)} \equiv -\Phi(x),$$

with the result that

$$p_1(x,0) = -(\eta^*\lambda/2\pi)\int_0^\infty \frac{d\xi}{(2\xi)^{\frac{1}{2}(2-\lambda)}(x-\xi)}, \tag{B 18}$$

found from (B 4).

The induced pressure gradient in (B 18) is exactly that which turns the flow in the wall layer. Hence, from (B 11), (B 18),

$$\beta(2x)^{-(\lambda-1)}/2(\lambda-1) = -(\eta^*\lambda/2\pi)\int_0^\infty \frac{d\xi}{(2\xi)^{\frac{1}{2}(2-\lambda)}(x-\xi)}$$

which with the transformation $m = \xi/x$ becomes

$$\beta(2x)^{-(\lambda-1)}/2(\lambda-1) = (\eta^*\lambda/2\pi)(2x)^{\frac{1}{2}(\lambda-2)}\int_0^\infty \frac{m^{-a}\,dm}{m-1}, \quad x>0$$

where $a = \frac{1}{2}(2-\lambda)$. It follows from the x-dependence of the result that $\lambda = \frac{4}{3}$, so that

$$\beta = \frac{4}{9}\frac{\eta^*}{\pi}\int_0^\infty \frac{m^{-\frac{1}{3}}\,dm}{m-1},$$

where the value of the integral is $\pi/\sqrt{3}$. Finally, from (B 13), we find

$$\eta^* = -\left(\frac{1}{3\beta}\right)^{\frac{1}{2}}\int_C^{C_0}\left[\left(\frac{C}{\sigma}\right)^{\frac{3}{2}}-1\right]^{\frac{1}{2}} d\sigma.$$

Hence, there are two equations for the constants β and η^*, which can then be obtained for any specified value $C > C_0$.

Finally, the wall shear and interaction pressure gradient in terms of the external co-ordinates are given by

$$\psi_{yy}(x,0) = Re^{\frac{1}{3}}(2x)^{-\frac{5}{6}}(\beta/C) \quad \text{and} \quad p(x) = Re^{-\frac{1}{3}}(2x)^{-\frac{1}{3}}(3\beta/2).$$

These results are to be compared with those found from the classical boundary-layer treatment of uniform flow past a flat plate with similarity type injection: $\psi_{yy}(x,0) = Re^{\frac{1}{2}}x^{-\frac{1}{2}}f''(0)$, $dp/dx = 0$. One notices that the wall shear has a different dependence on Reynolds number. Of course, it should be pointed out that, whereas the boundary-layer calculation is limited to injection values equal to at most $-C_0$, the interaction calculation is valid for only $C > C_0$. It would appear that, to obtain a smooth transition between the two regions of injection, one would have to abandon boundary-layer theory for values of C slightly less than C_0, in order to permit injectant layer thickness larger than $O(Re^{-\frac{1}{2}})$ and increasing in dimension toward $O(Re^{-\frac{1}{3}})$ as C passes through C_0. In other words, the singularity at $f(0) = -C_0$ is a product of the inadequacies of boundary-layer theory and not of reality.

REFERENCES

ACRIVOS, A. 1962 The asymptotic form of the laminar boundary layer mass transfer rate for large interfacial velocities. *J. Fluid Mech.* **12**, 337.

AROESTY, J. & COLE, J. D. 1967 Boundary layer flows with large injection rates. *The Rand Corp.*, RM 4620.

COLE, J. D. & AROESTY, J. 1967 The blowhard problem – inviscid flows with surface injection. *The Rand Corp.*, RM 5196.

EMMONS, H. W. & LEIGH, D. C. 1954 Tabulation of the Blasius function with blowing and suction. *Aero. Res. Counc. Current Paper* 157.

INGER, G. 1969 Strong blowing across a Couette–Poiseuille shear flow. *Phys. Fluids*, **13**, 1712.

KASSOY, D. R. 1970 On laminar boundary layer blowoff. *SIAM Appl. Math.* **17**, 24.

KUBOTA, T. & FERNANDEZ, F. 1968 Boundary layer flows with large injection and heat transfer. *AIAA J.* **6**, 22.

LIBBY, P. A. & KASSOY, D. R. 1970 The laminar boundary layer at an infinite swept stagnation line with large rates of injection, *AIAA J.* **8**, 1846.

LOCK, R. C. 1951 The velocity distribution in the laminar boundary layer between parallel streams. *Quart. J. Mech. Appl. Math.* **4**, 42.

MOORE, F. K. 1964 *Theory of Laminar Flows. High Speed Aerodynamics and Jet Propulsion*, vol. 4. Princeton University Press.

NICKEL, K. 1958 Mass transfer in boundary layer theory. *Arch. Rat. Mech. Anal.* **2**, 1.

PRETSCH, J. 1944 Analytic solution of the laminar boundary layer with asymptotic suction and injection. *ZAMM*, **24**, 264.

ROSENHEAD, L. (ed.) 1963 *Laminar Boundary Layers*. Oxford University Press.

SCHLICHTING, H. & BUSSMAN, K. 1943 Exacte Lösungen für die laminare Grenzschicht mit Absaugung and Ausblasen. *Schr. dtsch Akad Luftfahr*, **7**B.

STEWARTSON, K. 1964 Falkner–Skan equations for wakes. *AIAA J.* **2**, 1326.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.

WALLACE, J. & KEMP, N. 1969 Similarity solutions to the massive blowing problem. *AIAA J.* **7**, 1517.

WATSON, E. J. 1966 The equation of similar profiles in boundary layer theory with strong blowing. *Proc. Roy. Soc.* A **294**, 208.